A TRACING OF THE FRACTIONAL TEMPERATURE FIELD

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Abstract. This note is devoted to a study of L^q-tracing of the fractional temperature field $u(t,x)$ – the weak solution of the fractional heat equation $(\partial_t+(-\Delta_x)^\alpha)u(t,x)=g(t,x)$ in $L^p(\mathbb{R}^{1+n}_+)$ subject to the initial temperature $u(0,x)=f(x)$ in $L^p(\mathbb{R}^n)$.

1. Introduction

Directly continuing from [7, 12], we consider the fractional heat equation in the upper-half Euclidean space $\mathbb{R}^{1+n}_+=\mathbb{R}_+\times\mathbb{R}^n$ with $\mathbb{R}_+=(0,\infty)$ and $n\ge 1$:

$$\begin{cases}\big(\partial_t+(-\Delta_x)^\alpha\big)u(t,x)=g(t,x)\ \ \forall\ (t,x)\in\mathbb{R}^{1+n}_+;\\ u(0,x)=f(x)\ \ \forall\ x\in\mathbb{R}^n,\end{cases}\tag{1.1}$$

where $(-\Delta_x)^\alpha$ denotes the fractional $(0<\alpha<1)$ power of the spatial Laplacian that is determined by

$$(-\Delta_x)^\alpha u(\cdot,x)=\mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}u(\cdot,\xi))(x)\ \ \forall\ x\in\mathbb{R}^n$$

for which $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is its inverse. Specifically, we are interested in the trace of such a fractional temperature field (existing as the weak solution of (1.1))

$$u(t,x)=R_\alpha f(t,x)+S_\alpha g(t,x)$$

with

$$\begin{cases}R_\alpha f(t,x)=e^{-t(-\Delta_x)^\alpha}f(x)=\int_{\mathbb{R}^n}K_t^{(\alpha)}(x-y)f(y)dy;\\ S_\alpha g(t,x)=\int_0^t e^{-(t-s)(-\Delta_x)^\alpha}g(s,x)ds=\int_{\mathbb{R}^n}\big(\int_0^t K_{t-s}^{(\alpha)}(x-y)g(s,y)ds\big)dy,\end{cases}$$

where $K_t^{(\alpha)}(x)$ is the fractional heat kernel

$$K_t^{(\alpha)}(x)\equiv(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{ix\cdot y-t|y|^{2\alpha}}dy\ \ \forall\ (t,x)\in\mathbb{R}^{1+n}_+$$

whose endpoint $\alpha=1$ and middle-point $\alpha=1/2$ lead to the heat kernel and Poisson kernel:

$$K_t^{(1)}(x)=(4\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}\quad\&\quad K_t^{(\frac12)}(x)=\pi^{-\frac{n+1}{2}}\Gamma\Big(\frac{n+1}{2}\Big)\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}$$

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with $\Gamma(\cdot)$ being the classical gamma function. Although there is no explicit formula for $K_t^{(\alpha)}(x)$ under $\alpha \in (0,1) \setminus \{1/2\}$ (cf. [8, 10, 13, 14, 15, 16, 18, 19, 17, 22]), the following estimates are not only valid but also practical (cf. [3, 4, 5, 9, 20]):

$$\begin{cases} K_t^{(\alpha)}(x) \approx \min\{t^{-\frac{n}{2\alpha}}, t|x|^{-(n+2\alpha)}\} \approx \frac{t}{(t^{\frac{1}{2\alpha}}+|x|)^{n+2\alpha}} \ \ \forall\ (t,x) \in \mathbb{R}^{1+n}_+; \\ \int_{\mathbb{R}^n} K_t^{(\alpha)}(x)dx = 1 \ \ \forall\ t \in (0,\infty). \end{cases}$$

As explored in [7, 12], the regularity of $u(t,x)$ sheds some light on the traces/restrictions of $R_\alpha f(t,x)$ and $S_\alpha g(t,x)$ to subsets of $\mathbb{R}^{1+n}_+$ of $(1+n)$-dimensional Lebesgue measure zero. Here $f(x)$ and $g(t,x)$ are arbitrary functions of the usual Lebesgue classes $L^p(\mathbb{R}^n)$ and $L^p(\mathbb{R}^{1+n}_+)$, respectively. In order to characterize the traces of $R_\alpha f(t,x)$ and $S_\alpha g(t,x)$ on a given compact exceptional set $K \subset \mathbb{R}^{1+n}_+$, we investigate nonnegative Radon measures supported on K such that under $1 < p,q < \infty$ the mapping $R_\alpha : L^p(\mathbb{R}^n) \mapsto L^q_\mu(\mathbb{R}^{1+n}_+)$ and $S_\alpha : L^p(\mathbb{R}^{1+n}_+) \mapsto L^q_\mu(\mathbb{R}^{1+n}_+)$ are continuous - namely -

$$\left(\int_{\mathbb{R}^{1+n}_+} |R_\alpha f(t,x)|^q \, d\mu(t,x)\right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{1.2}$$

and

$$\left(\int_{\mathbb{R}^{1+n}_+} |S_\alpha g(t,x)|^q \, d\mu(t,x)\right)^{\frac{1}{q}} \lesssim \|g\|_{L^p(\mathbb{R}^{1+n}_+)}, \tag{1.3}$$

where the symbol $A \lesssim B$ means $A \le cB$ for a positive constant c - moreover - $A \approx B$ stands for both $A \lesssim B$ and $B \lesssim A$.

A careful examination of (1.2) and (1.3) indicates that they can be naturally unified as:

$$\left(\int_{\mathbb{R}^{1+n}_+} |T_\alpha h|^q \, d\mu\right)^{\frac{1}{q}} \lesssim \|h\|_{L^p(\mathbb{X})} = \begin{cases} \|f\|_{L^p(\mathbb{R}^n)} \text{ as } (T_\alpha, h, \mathbb{X}) = (R_\alpha, f, \mathbb{R}^n); \\ \|g\|_{L^p(\mathbb{R}^{1+n}_+)} \text{ as } (T_\alpha, h, \mathbb{X}) = (S_\alpha, g, \mathbb{R}^{1+n}_+). \end{cases} \tag{1.4}$$

Describing such a measure μ on $\mathbb{R}^{1+n}_+$ depends on a concept of the induced capacity. For a compact set $K \subset \mathbb{R}^{1+n}_+$ let

$$C_p^{(T_\alpha)}(K) = \inf\{\|h\|^p_{L^p(\mathbb{X})} : h \ge 0 \ \&\ T_\alpha h \ge \mathbf{1}_K\},$$

where $\mathbf{1}_K$ is the characteristic function of K. Then, for an open subset O of $\mathbb{R}^{1+n}_+$ let

$$C_p^{(T_\alpha)}(O) = \sup\{C_p^{(T_\alpha)}(K) : \text{compact}\, K \subset O\},$$

and hence for any set $E \subset \mathbb{R}^{1+n}_+$ let

$$C_p^{(T_\alpha)}(E) = \inf\{C_p^{(T_\alpha)}(O) : \text{open}\, O \supset E\}.$$

According to [7, 12], if

$$B_r^{(\alpha)}(t_0,x_0) \equiv \{(t,x) \in \mathbb{R}^{1+n}_+ : r^{2\alpha} < t - t_0 < 2r^{2\alpha} \ \&\ |x - x_0| < r\}$$

stands for the parabolic ball with centre $(t_0,x_0)\in\mathbb{R}^{1+n}_+$ and radius $r>0$, then

$$C_p^{(T_\alpha)}\big(B_r^{(\alpha)}(t_0,x_0)\big)\approx\begin{cases} r^n \text{ as } T_\alpha=R_\alpha;\\ r^{n+2\alpha(1-p)} \text{ as } T_\alpha=S_\alpha \ \& \ 1<p<1+\frac{n}{2\alpha}.\end{cases}\tag{1.5}$$

Below is a tracing principle for the fractional heat equation (1.1).

Theorem 1.1. *Let $0<\alpha<1$ and $1<p<1+\frac{n}{2\alpha}$. Then*

$$(1.4)\Leftrightarrow\begin{cases}\sup\left\{\frac{\mu\left(B_r^{(\alpha)}(t_0,x_0)\right)}{\left(C_p^{(T_\alpha)}\left(B_r^{(\alpha)}(t_0,x_0)\right)\right)^{q/p}}:\ (r,t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n\right\}<\infty \ \textit{as}\ p<q;\\ \sup\left\{\frac{\mu(K)}{C_p^{(T_\alpha)}(K)}:\ \textit{compact}\ K\subset\mathbb{R}^{1+n}_+\right\}<\infty\ \textit{as}\ p=q;\\ \int_{\mathbb{R}^{1+n}_+}\left(\int_0^\infty\left(\frac{\mu(B_r^{(\alpha)}(t_0,x_0))}{C_p^{(T_\alpha)}(B_r^{(\alpha)}(t_0,x_0))}\right)^{\frac{1}{p-1}}\frac{dr}{r}\right)^{\frac{q(p-1)}{p-q}}d\mu(t_0,x_0)<\infty\ \textit{as}\ p>q.\end{cases}$$

Here, it should be noted that R_α-case of Theorem 1.1 under $p\le q$ has been treated in [7, Theorems 3.2-3.3]. Of course, the remaining cases of Theorem 1.1 are new. Perhaps, it is worth to point out that under $p=q$,

$$\sup\left\{\frac{\mu(K)}{C_p^{(T_\alpha)}(K)}:\ \text{compact } K\subset\mathbb{R}^{1+n}_+\right\}<\infty$$

implies

$$\sup\left\{\frac{\mu(B_r^\alpha(t_0,x_0))}{C_p^{(T_\alpha)}(B_r^\alpha(t_0,x_0))}:\ B_r^\alpha(t_0,x_0)\subset\mathbb{R}^{1+n}_+\right\}<\infty$$

but not conversely in general - [1, Theorem 4(ii)] and its argument might be helpful to produce a ball-based sufficient condition for (1.4) to hold. Upon $d\mu=dtdx$ in S_α-case of Theorem 1.1 we have $\mu(B_r^{(\alpha)}(t_0,x_0))\approx r^{n+2\alpha}$, thereby finding that (cf. [21, Theorem 1.4]) for $g\in L^p(\mathbb{R}^{1+n}_+)$ one has

$$\|S_\alpha g\|_{L^{\widetilde{q}}(\mathbb{R}^{1+n}_+)}\lesssim\|g\|_{L^p(\mathbb{R}^{1+n}_+)}\quad\text{where } \widetilde{q}=p\left(1+\frac{2\alpha p}{n+2\alpha-2\alpha p}\right)>p.$$

Although R_α and S_α behave similarly, the argument for Theorem 1.1 will be still split into two parts - one for R_α in Section 2 and another one for S_α in Section 3 - this is because the subtle difference between R_α and S_α can be seen clearly from such a splitting arrangement.

2. R_α's tracing

In this section we verify Theorem 1.1 for $T_\alpha=R_\alpha$. To do so, we need three lemmas as seen below.

The first is about the dual representation of $C_p^{(R_\alpha)}(K)$ for a given compact set $K\subset\mathbb{R}^{1+n}_+$.

Lemma 2.1. *Let $\mathcal{U}^+(K)$ be the class of all nonnegative Radon measures μ with compact support $K\subset\mathbb{R}^{1+n}_+$ and the total variation $\|\mu\|$. Then*

$$C_p^{(R_\alpha)}(K)=\sup\left\{\|\mu\|^p:\quad \mu\in\mathcal{U}^+(K)\ \&\ \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^{1+n}_+}K_t^\alpha(x-y)\,d\mu(t,y)\Big)^{\frac{p}{p-1}}dx\le 1\right\}.$$

Proof. Note that

$$\int_{\mathbb{R}^{1+n}_+} R_\alpha f(t,x)h(t,x)dtdx = \int_{\mathbb{R}^n} f(x)\Big(\int_{\mathbb{R}^{1+n}_+} K_t^{(\alpha)}(x-y)h(t,y)dtdy\Big)dx$$

holds for all $(f,h)\in C_0^\infty(\mathbb{R}^n)\times C_0^\infty(\mathbb{R}^{1+n}_+)$ where $C_0^\infty(\mathbb{X})$ stands for the class of infinitely differentiable functions with compact support in $\mathbb{X}=\mathbb{R}^n$ or $\mathbb{R}^{1+n}_+$. Thus, the adjoint operator of R_α is defined by

$$(R_\alpha^* h)(x) = \int_{\mathbb{R}^{1+n}_+} K_t^{(\alpha)}(x-y)h(t,y)dtdy \ \ \forall\ h\in C_0^\infty(\mathbb{R}^{1+n}_+).$$

For any nonnegative Radon measure μ in $\mathbb{R}^{1+n}_+$ and a continuous function f with a compact support in $\mathbb{R}^n$, one has

$$\left|\int_{\mathbb{R}^{1+n}_+} R_\alpha f d\mu\right| \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}\|\mu\|.$$

Therefore, the Riesz representation theorem yields a Borel measure ν on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^{1+n}_+} R_\alpha f d\mu = \int_{\mathbb{R}^n} f d\nu \ \ \forall\ f\ge 0.$$

This means that $\nu = R_\alpha^*\mu$ can be defined by

$$R_\alpha^*\mu(x) = \int_{\mathbb{R}^{1+n}_+} K_t^{(\alpha)}(x-y)d\mu(t,y).$$

According to [7, Proposition 1], one gets

$$C_p^{(R_\alpha)}(K) = \sup\Big\{\|\mu\|^p:\ \mu\in\mathcal{U}^+(K)\ \&\ \|R_\alpha^*\mu\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)}\le 1\Big\}.$$

□

The second is about L^p-boundedness of the fractional maximal operator of parabolic type.

Lemma 2.2. *For a nonnegative Radon measure μ on $\mathbb{R}^{1+n}_+$ let*

$$M_\alpha\mu(x) = \sup_{r>0} r^{-n}\mu\big(B_r^{(\alpha)}(r^{2\alpha},x)\big)$$

be its fractional parabolic maximal function. Then

$$\|M_\alpha\mu\|_{L^p(\mathbb{R}^n)} \approx \|R_\alpha^*\mu\|_{L^p(\mathbb{R}^n)} \ \ \forall\ \ p\in(1,\infty).$$

Proof. A straightforward estimation with $x\in\mathbb{R}^n$ and $R_\alpha^*\mu(x)$ gives

$$R_\alpha^*\mu(x) \gtrsim \int_{B_r^{(\alpha)}(r^{2\alpha},x)} \frac{t}{(t^{\frac{1}{2\alpha}}+|x-y|)^{n+2\alpha}}d\mu(t,y) \gtrsim \frac{\mu(B_r^{(\alpha)}(r^{2\alpha},x))}{r^n} \ \ \forall\ \ r>0,$$

whence

$$R_\alpha^\alpha\mu(x) \gtrsim M_\alpha\mu(x).$$

This implies

$$\|M_\alpha\mu\|_{L^p(\mathbb{R}^n)} \lesssim \|R_\alpha^*\mu\|_{L^p(\mathbb{R}^n)}.$$

To prove the converse inequality, we slightly modify [2, (3.6.1)] to get two constants $a>1$ and $b>0$ such that for any $\lambda>0$ and $0<\varepsilon\le 1$, one has the following good-λ inequality

$$\begin{aligned}|\{x\in\mathbb{R}^n:\ R^*_\alpha\mu(x)>a\lambda\}| &\le b\varepsilon^{\frac{n+2\alpha}{n}}|\{x\in\mathbb{R}^n:\ R^*_\alpha\mu(x)>\lambda\}|\\ &\quad+|\{x\in\mathbb{R}^n:\ M_\alpha\mu(x)>\varepsilon\lambda\}|.\end{aligned}\tag{2.1}$$

Inspired by [2, Theorem 3.6.1], we proceed the proof by using (2.1). Multiplying (2.1) by λ^{p-1} and integrating in λ, we have for any $\gamma>0$,

$$\begin{aligned}\int_0^\gamma |\{x\in\mathbb{R}^n:\ R^*_\alpha\mu(x)>a\lambda\}|\lambda^{p-1}d\lambda &\le b\varepsilon^{\frac{n+2\alpha}{n}}\int_0^\gamma |\{x\in\mathbb{R}^n:\ R^*_\alpha\mu(x)>\lambda\}|\lambda^{p-1}d\lambda\\ &\quad+\int_0^\gamma |\{x\in\mathbb{R}^n:\ M_\alpha\mu(x)>\varepsilon\lambda\}|\lambda^{p-1}d\lambda.\end{aligned}$$

An equivalent formulation of the above inequality is

$$\begin{aligned}a^{-p}\int_0^{a\gamma} |\{x\in\mathbb{R}^n:\ R^*_\alpha\mu(x)>a\lambda\}|\lambda^{p-1}d\lambda &\le b\varepsilon^{\frac{n+2\alpha}{n}}\int_0^\gamma |\{x\in\mathbb{R}^n:\ R^*_\alpha\mu(x)>\lambda\}|\lambda^{p-1}d\lambda\\ &\quad+\varepsilon^{-p}\int_0^{\varepsilon\gamma} |\{x\in\mathbb{R}^n:\ M_\alpha\mu(x)>\varepsilon\lambda\}|\lambda^{p-1}d\lambda.\end{aligned}$$

Let ε be so small that $b\varepsilon^{\frac{n+2\alpha}{n}}\le\frac12 a^{-p}$ and $\gamma\to\infty$. Then

$$a^{-p}\int_{\mathbb{R}^n}(R^*_\alpha\mu(x))^p dx\le 2\varepsilon^{-p}\int_{\mathbb{R}^n}(M_\alpha\mu(x))^p dx.$$

That is

$$\|M_\alpha\mu\|_{L^p(\mathbb{R}^n)}\gtrsim\|R^*_\alpha\mu\|_{L^p(\mathbb{R}^n)}.$$

□

The third is about the Hedberg-Wolff potential for R_α:

$$P^R_{\alpha p}\mu(t,x)=\int_0^\infty\left(\frac{\mu(B^{(\alpha)}_r(t,x))}{r^n}\right)^{p'-1}\frac{dr}{r}\quad\forall\ (t,x)\in\mathbb{R}^{1+n}_+.$$

Lemma 2.3. *Let $1<p<\infty$, $p'=\frac{p}{p-1}$, and μ be a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$. Then*

$$\|R^*_\alpha\mu\|^{p'}_{L^{p'}(\mathbb{R}^n)}\approx\int_{\mathbb{R}^{1+n}_+}P^R_{\alpha p}\mu\,d\mu.$$

Proof. Below is a two-fold argument.

Part 1. The first task is to show

$$\|R^*_\alpha\mu\|^{p'}_{L^{p'}(\mathbb{R}^n)}\lesssim\int_{\mathbb{R}^{1+n}_+}P^R_{\alpha p}\mu\,d\mu.$$

Note first that

$$\frac{\mu(B^{(\alpha)}_r(r^{2\alpha},x))}{r^n}\approx\left(\int_r^{2r}\left(\frac{\mu(B^{(\alpha)}_s(s^{2\alpha},x))}{s^n}\right)^{p'}\frac{ds}{s}\right)^{\frac{1}{p'}}\lesssim\left(\int_0^{\infty}\left(\frac{\mu(B^{(\alpha)}_s(s^{2\alpha},x))}{s^n}\right)^{p'}\frac{ds}{s}\right)^{\frac{1}{p'}}.$$

Therefore, one has

$$M_\alpha\mu(x) \lesssim \left(\int_0^\infty \left(\frac{\mu(B_s^{(\alpha)}(s^{2\alpha},x))}{s^n}\right)^{p'}\frac{ds}{s}\right)^{\frac{1}{p'}}.$$

By Lemma 2.2, it is sufficient to verify

$$\int_{\mathbb{R}^n}\int_0^\infty \left(\frac{\mu(B_r^{(\alpha)}(r^{2\alpha},x))}{r^n}\right)^{p'}\frac{dr}{r}dx \lesssim \int_{\mathbb{R}^{1+n}_+}\int_0^\infty \left(\frac{\mu(B_r^{(\alpha)}(t,x))}{r^n}\right)^{p'-1}\frac{dr}{r}d\mu.$$

Using the Fubini theorem, one has

$$\int_{\mathbb{R}^n}\int_0^\infty \left(\frac{\mu(B_r^{(\alpha)}(r^{2\alpha},x))}{r^n}\right)^{p'}\frac{dr}{r}dx = \int_0^\infty\int_{\mathbb{R}^n}\frac{\mu(B_r^{(\alpha)}(r^{2\alpha},x))^{p'}}{r^{np'}}dx\frac{dr}{r}.$$

A further application of Fubini's theorem yields

$$\begin{aligned}\int_{\mathbb{R}^n}\mu(B_r^{(\alpha)}(r^{2\alpha},x))^{p'}dx &\lesssim \int_{B_r^{(\alpha)}(r^{2\alpha},x)}\int_{\mathbb{R}^n}\mu(B_r^{(\alpha)}(r^{2\alpha},x))^{p'-1}dxd\mu\\ &\lesssim \int_{B_r^{(\alpha)}(r^{2\alpha},x)}\int_{\mathbb{R}^n}\mu(B_r^{(\alpha)}(r^{2\alpha},y))^{p'-1}dxd\mu\\ &\lesssim r^n\int_{B_r^{(\alpha)}(r^{2\alpha},x)}\mu(B_r^{(\alpha)}(r^{2\alpha},y))^{p'-1}d\mu.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^n}\frac{\mu(B_r^{(\alpha)}(r^{2\alpha},x))^{p'}}{r^{np'}}dx\frac{dr}{r} &\approx \int_0^\infty\int_{B_r^{(\alpha)}(r^{2\alpha},x)}\frac{\mu(B_r^{(\alpha)}(r^{2\alpha},y))^{p'-1}}{r^{n(p'-1)}}d\mu\frac{dr}{r}\\ &\approx \int_{B_r^{(\alpha)}(r^{2\alpha},x)}\int_0^\infty\left(\frac{\mu(B_r^{(\alpha)}(r^{2\alpha},y))}{r^n}\right)^{p'-1}\frac{dr}{r}d\mu\\ &\lesssim \int_{\mathbb{R}^{1+n}_+}\left(\int_0^\infty\left(\frac{\mu(B_r^{(\alpha)}(t,x))}{r^n}\right)^{p'-1}\frac{dr}{r}\right)d\mu(t,x),\end{aligned}$$

as desired.

Part 2. The second task is to prove

$$\|R^*_\alpha\mu\|^{p'}_{L^{p'}(\mathbb{R}^n)} \gtrsim \int_{\mathbb{R}^{1+n}_+}P^R_{\alpha p}\mu\, d\mu.$$

Since

$$\begin{aligned}\|R^*_\alpha\mu\|^{p'}_{L^{p'}(\mathbb{R}^n,\,dx)} &= \int_{\mathbb{R}^n}\left(R^*_\alpha\mu(x)\right)^{p'-1}(R^*_\alpha\mu(x))dx\\ &= \int_{\mathbb{R}^{1+n}_+}\int_{\mathbb{R}^n}(R^*_\alpha\mu(x))^{p'-1}K_t^{(\alpha)}(x-y)dx\,d\mu(t,y).\end{aligned}$$

Upon writing

$$K(t,x) = \int_{\mathbb{R}^n}(R^*_\alpha\mu(x))^{p'-1}K_t^{(\alpha)}(x-y)dx$$

and

$$B(x,2^{-m})=\{y\in\mathbb{R}^n:\ |x-y|<2^{-m}\ \&\ (2^{-m})^{2\alpha}<t<2(2^{-m})^{2\alpha}\}\ \forall\ m\in\mathbb{Z}\equiv\{0,\pm1,\pm2,...\},$$

we obtain

$$\begin{aligned}
K(t,x) &\approx \int_{\mathbb{R}^n}\frac{t}{(t^{\frac{1}{2\alpha}}+|x-y|)^{n+2\alpha}}\left(\int_{\mathbb{R}^{1+n}_+}\frac{s}{(s^{\frac{1}{2\alpha}}+|y-z|)^{n+2\alpha}}d\mu\right)^{p'-1}dy\\
&\gtrsim \sum_{m\in\mathbb{Z}}\int_{B(x,2^{-m})}t^{-\frac{n}{2\alpha}}\left(\int_{B^{(\alpha)}_{2^{-m}}(t,x)}s^{-\frac{n}{2\alpha}}d\mu\right)^{p'-1}dy\\
&\gtrsim \sum_{m\in\mathbb{Z}}\int_{B(x,2^{-m})}2^{mn}\left(\frac{\mu(B^{(\alpha)}_{2^{-m}}(t,x))}{2^{-m}}\right)^{p'-1}dy\\
&\gtrsim \int_0^\infty\frac{1}{r^n}\int_{B(x,2^{-m})}2^{mn}\left(\frac{\mu(B^{(\alpha)}_{2^{-m}}(t,x))}{2^{-m}}\right)^{p'-1}dy\frac{dr}{r}\\
&\gtrsim \int_0^\infty\left(\frac{\mu(B^{(\alpha)}_{r}(t,x))}{r^n}\right)^{p'-1}\frac{dr}{r},
\end{aligned}$$

thereby reaching the required inequality.

□

Now, Theorem 1.1 with $T_\alpha=R_\alpha$ is contained in the following result.

Theorem 2.4. *For a nonnegative Radon measure μ on $\mathbb{R}^{1+n}_+$ and $\lambda>0$ set*

$$C_R(\mu;\lambda)=\inf\left\{C_p^{(R_\alpha)}(K):\ \text{compact } K\subset\mathbb{R}^{1+n}_+\ \&\ \mu(K)\ge\lambda\right\}.$$

(1) *If $1<p<q<\infty$ then*

$$(1.2)\Leftrightarrow\sup_{\lambda>0}\frac{\lambda^{\frac{p}{q}}}{C_R(\mu;\lambda)}<\infty\Leftrightarrow\sup_{(r,t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n}\frac{\mu(B^{(\alpha)}_r(t_0,x_0))}{r^{\frac{nq}{p}}}<\infty.$$

(2) *If $1<p=q<\infty$ then*

$$(1.2)\Leftrightarrow\sup_{\lambda>0}\frac{\lambda}{C_R(\mu;\lambda)}<\infty\quad\left(\Rightarrow\sup_{(r,t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n}\frac{\mu(B^{(\alpha)}_r(t_0,x_0))}{r^n}<\infty\right).$$

(3) *$1<q<p<\infty$ then*

$$(1.2)\Leftrightarrow\int_0^\infty\left(\frac{\lambda^{\frac{p}{q}}}{C_R(\mu;\lambda)}\right)^{\frac{q}{p-q}}\frac{d\lambda}{\lambda}<\infty\Leftrightarrow P^R_{\alpha p}\mu\in L^{q(p-1)/(p-q)}_\mu(\mathbb{R}^{1+n}_+).$$

Proof. Since (1), (2) and the left equivalence of (3) are contained in [7, Theorems 3.2&3.3] whose proofs depend on Lemma 2.1, it is enough to check the right equivalence of (3). Our approach is a fractional heat potential analogue of the Riesz potential treatment carried in [6, Theorem 2.1].

Step 1. We show

$$(1.2)\Rightarrow P^R_{\alpha p}\mu\in L^{q(p-1)/(p-q)}_\mu(\mathbb{R}^{1+n}_+).$$

To do so, we first denote by $Q_l^{(\alpha)}$ the α-dyadic cube with side length $l\equiv l(Q_l^{(\alpha)})$ and corners in the set $\{l^{2\alpha}\mathbb{Z}_+, l\mathbb{Z}^n\}$ with $\mathbb{Z}_+=\{0,1,2,\cdots\}$ - namely -

$$Q_l^{(\alpha)}\equiv\{[k_0l^{2\alpha},(k_0+1)l^{2\alpha})\times[k_1l,(k_1+1)l)\times\cdots\times[k_nl,(k_n+1)l)\}\ \text{ as }\ k_0\in\mathbb{Z}_+\ \&\ k_i\in\mathbb{Z}$$

for $i=1,2,\cdots,n$. Next, we introduce the following fractional heat Hedberg-Wolff potential generated by $\mathcal{D}^\alpha$ - the family of all the above-defined α-dyadic cubes in $\mathbb{R}^{1+n}_+$:

$$P^{d,R}_{\alpha p}\mu(t,x)=\sum_{Q_l^{(\alpha)}\in\mathcal{D}^\alpha}\left(\frac{\mu(Q_l^{(\alpha)})}{l^n}\right)^{p'-1}\mathbf{1}_{Q_l^{(\alpha)}}(t,x)$$

and then prove

$$(1.2)\Rightarrow\int_{\mathbb{R}^{1+n}_+}(P^{d,R}_{\alpha p}\mu(t,x))^{\frac{q(p-1)}{(p-q)}}d\mu(t,x)<\infty. \tag{2.2}$$

Indeed, by duality, (1.2) is equivalent to the following inequality

$$\|R^*_\alpha(\mathbf{g}d\mu)\|^{p'}_{L^{p'}(\mathbb{R}^n)}\lesssim\|\mathbf{g}\|^{p'}_{L^{q'}_\mu(\mathbb{R}^{1+n}_+)}\quad\forall\ \mathbf{g}\in L_\mu^{q'=\frac{q}{q-1}}(\mathbb{R}^{1+n}_+).$$

It is easy to check that Lemma 2.3 is also true with $P^{d,R}_{\alpha p}\mu$ in place of $P^R_{\alpha p}\mu$ and $\mathbf{g}d\mu$ in place of $d\mu$. So, one has

$$\|R^*_\alpha(\mathbf{g}d\mu)\|^{p'}_{L^{p'}(\mathbb{R}^n)}\gtrsim\int_{\mathbb{R}^{1+n}_+}P^{d,R}_{\alpha p}(\mathbf{g}d\mu)(t,x)\mathbf{g}(t,x)d\mu(t,x)\gtrsim\sum_{Q_l^{(\alpha)}}\left(\frac{\int_{Q_l^{(\alpha)}}\mathbf{g}(t,x)d\mu(t,x)}{l^n}\right)^{p'}l^n.$$

Consequently,

$$\sum_{Q_l^{(\alpha)}}\left(\frac{\int_{Q_l^{(\alpha)}}\mathbf{g}(t,x)d\mu(t,x)}{l^n}\right)^{p'}l^n\lesssim\|\mathbf{g}\|^{p'}_{L^{q'}_\mu(\mathbb{R}^{1+n}_+)}. \tag{2.3}$$

Upon setting

$$\lambda_{Q_l^{(\alpha)}}=\left(\frac{\mu(Q_l^{(\alpha)})}{l^n}\right)^{p'}l^n,$$

one finds that (2.3) is equivalent to

$$\sum_{Q_l^{(\alpha)}}\lambda_{Q_l^{(\alpha)}}\left(\frac{\int_{Q_l^{(\alpha)}}\mathbf{g}\,d\mu}{\mu(Q_l^{(\alpha)})}\right)^{p'}\lesssim\|\mathbf{g}\|^{p'}_{L^{q'}_\mu(\mathbb{R}^{1+n}_+)}.$$

Define the following dyadic Hardy-Littlewood maximal function

$$M^d_\mu h(t,x)=\sup_{(t,x)\in Q^{(\alpha)}}\frac{1}{\mu(Q^{(\alpha)})}\int_{Q^{(\alpha)}}|h(s,y)|d\mu(s,y)\ \ \forall\ Q^{(\alpha)}\in\mathcal{D}^\alpha.$$

Then M^d_μ is bounded on $L^p_\mu(\mathbb{R}^{1+n}_+)$ for $1<p<\infty$. Write

$$\mathbf{g}(t,x)=(M^d_\mu h)^{\frac{1}{p'}}(t,x)\text{ under }0\le h\in L^{q'/p'}_\mu(\mathbb{R}^{1+n}_+).$$

It is easy to check that

$$\left(\frac{\int_{Q_l^{(\alpha)}}\mathbf{g}(t,x)d\mu(t,x)}{\mu(Q_l^{(\alpha)})}\right)^{p'}\gtrsim\frac{\int_Q h(t,x)d\mu(t,x)}{\mu(Q_l^{(\alpha)})}$$

and so that

$$\|\mathbf{g}\|^{p'}_{L^{q'}_\mu(\mathbb{R}^{1+n}_+)}\lesssim\|h\|_{L^{q'/p'}_\mu(\mathbb{R}^{1+n}_+)}.$$

This in turn implies

$$\sum_{Q_l^{(\alpha)}}\lambda_{Q_l^{(\alpha)}}\frac{\int_{Q_l^{(\alpha)}}h(t,x)d\mu(t,x)}{\mu(Q_l^{(\alpha)})}\lesssim\|h\|_{L^{q'/p'}_\mu(\mathbb{R}^{1+n}_+)}\ \forall\ h\in L^{q'/p'}_\mu(\mathbb{R}^{1+n}_+),$$

and thus via duality

$$\sum_{Q_l^{(\alpha)}}\frac{\lambda_{Q_l^{(\alpha)}}}{\mu(Q_l^{(\alpha)})}\mathbf{1}_{Q_l^{(\alpha)}}\in L_\mu^{\frac{q'}{q'-p'}}(\mathbb{R}^{1+n}_+),$$

namely,

$$\sum_{Q_l^{(\alpha)}}\left(\frac{\mu(Q_l^{(\alpha)})}{l^n}\right)^{p'-1}\mathbf{1}_{Q_l^{(\alpha)}}\in L_\mu^{\frac{q(p-1)}{p-q}}(\mathbb{R}^{1+n}_+),$$

which yields (2.2).

Next, set

$$P^{d,\tau,R}_{\alpha p}\mu(t,x)=\sum_{Q_l^{(\alpha)}\in\mathcal{D}^\alpha_\tau}\left(\frac{\mu(Q_l^{(\alpha)})}{l^n}\right)^{p'-1}\mathbf{1}_{Q_l^{(\alpha)}}(t,x)\ \ \&\ \ \mathcal{D}^\alpha_\tau=\mathcal{D}^\alpha+\tau=\{Q_l^{(\alpha)'}+\tau\}_{Q_l^{(\alpha)'}\in\mathcal{D}^\alpha},$$

where $Q_l^{(\alpha)}+\tau=\{(t,x)+\tau:\ (t,x)\in Q_l^{(\alpha)}\}$ means the $\mathbb{R}^{1+n}_+\ni\tau$-shift of $Q_l^{(\alpha)}$. Then (2.2) implies

$$\sup_{\tau\in\mathbb{R}^{1+n}_+}\int_{\mathbb{R}^{1+n}_+}\left(P^{d,\tau,R}_{\alpha p}\mu(t,x)\right)^{\frac{q(p-1)}{(p-q)}}d\mu(t,x)<\infty. \tag{2.4}$$

Now, it remains to prove

$$P^R_{\alpha p}\mu\in L_\mu^{q(p-1)/(p-q)}(\mathbb{R}^{1+n}_+).$$

Two situations are considered in the sequel.

Case 1.1. μ is a doubling measure. In this case, $P^R_{\alpha p}\mu\in L_\mu^{q(p-1)/(p-q)}(\mathbb{R}^{1+n}_+)$ is a by-product of (2.2) and the following observation

$$P^R_{\alpha p}\mu(t,x)\lesssim\sum_{Q_l^{(\alpha)}}\left(\frac{\mu(Q_l^{(\alpha)*})}{l^n}\right)^{p'-1}\mathbf{1}_{Q_l^{(\alpha)}}(t,x),$$

where $Q_l^{(\alpha)*}$ is the cube with the same center as $Q_l^{(\alpha)}$ and side length two times as $Q_l^{(\alpha)}$.

Case 1.2. μ is a possibly non-doubling measure. For any $\rho>0$, write

$$P^R_{\alpha p,\rho}\mu(t,x)=\int_0^\rho\left(\frac{\mu(B_r^{(\alpha)}(t,x))}{r^n}\right)^{p'-1}\frac{dr}{r}.$$

Then

$$P^R_{\alpha p,\rho}\mu(t,x)\lesssim \rho^{-(n+1)}\int_{|\tau|\lesssim\rho}P^{d,\tau,R}_{\alpha p}\mu(t,x)d\tau.$$

In fact, for a fixed $x\in\mathbb{R}^n$ and $\rho>0$ with $2^{i-1}\eta\le\rho<2^i\eta$ (where $i\in\mathbb{Z}$ and $\eta>0$ will be determined later) one has

$$P^R_{\alpha p,\rho}\mu(t,x)\lesssim\sum_{j=-\infty}^{i}\left(\frac{\mu(B^{(\alpha)}_{2^j\eta}(t,x))}{(2^j\eta)^n}\right)^{p'-1}.$$

For $j\le i$, let $Q^{(\alpha)}_{l,j}$ be a cube centred at x with $2^{j-1}<l\le 2^j$. Then $B^{(\alpha)}_{2^j\eta}(t,x)\subseteq Q^{(\alpha)}_{l,j}$ for sufficiently small η. Assume not only that E is the set of all points $\tau\in\mathbb{R}^{1+n}_+$ enjoying $|\tau|\lesssim\rho$ with $|E|$ being the $(1+n)$-dimensional Lebesgue measure, but also that there exists $Q^{(\alpha),\tau}_l\in\mathcal{D}^\alpha_\tau$ satisfying $l=2^{j+1}$ and $Q^{(\alpha)}_{l,j}\subseteq Q^{(\alpha),\tau}_l$. A geometric consideration produces a dimensional constant $c(n)>0$ such that $|E|\ge c(n)\rho^{n+1}\ \forall\ j\le i$. Consequently, one has

$$\begin{aligned}\mu(B^{(\alpha)}_{2^j\eta}(t,x))^{p'-1}&\lesssim|E|^{-1}\int_E\sum_{l=2^{j+1}}\mu(Q^{(\alpha),\tau}_l)^{p'-1}\mathbf{1}_{Q^{(\alpha),\tau}_l}(t,x)d\tau\\&\lesssim\rho^{-(n+1)}\int_{|\tau|\lesssim\rho}\sum_{l=2^{j+1}}\mu(Q^{(\alpha),\tau}_l)^{p'-1}\mathbf{1}_{Q^{(\alpha),\tau}_l}(t,x)d\tau,\end{aligned}$$

and so that

$$\begin{aligned}P^R_{\alpha p,\rho}\mu(t,x)&\lesssim\rho^{-(n+1)}\int_{|\tau|\lesssim\rho}\sum_{j=-\infty}^{i}\sum_{l=2^{j+1}}\left(\frac{\mu(Q^{(\alpha),\tau}_l)}{(2^j\eta)^n}\right)^{p'-1}\mathbf{1}_{Q^{(\alpha),\tau}_l}(t,x)ds\\&\lesssim\rho^{-(n+1)}\int_{|\tau|\lesssim\rho}P^{d,\tau,R}_{\alpha p}\mu(t,x)d\tau,\end{aligned}$$

whence reaching (2).

From (2), the Hölder inequality and Fubini's theorem it follows that

$$\begin{aligned}&\int_{\mathbb{R}^{1+n}_+}\left(P^R_{\alpha p,\rho}\mu(t,x)\right)^{\frac{q(p-1)}{p-q}}d\mu(t,x)\\&\lesssim\int_{\mathbb{R}^{1+n}_+}\left(\rho^{-(n+1)}\left(\int_{|\tau|\le C\rho}\left(P^{d,\tau,R}_{\alpha p}\mu\right)^{\frac{q(p-1)}{p-q}}d\tau\right)^{\frac{p-q}{q(p-1)}}\left(\int_{|\tau|\lesssim\rho}d\tau\right)^{1-\frac{p-q}{q(p-1)}}\right)^{\frac{q(p-1)}{p-q}}d\mu\\&\lesssim\rho^{-(n+1)}\int_{|\tau|\lesssim\rho}\left(\int_{\mathbb{R}^{1+n}_+}\left(P^{d,\tau,R}_{\alpha p}\mu\right)^{\frac{q(p-1)}{p-q}}d\mu\right)d\tau\\&\le\kappa(n),\end{aligned}$$

where the last constant $\kappa(n)$ is independent of ρ. This clearly produces

$$P^R_{\alpha p}\mu\in L^{q(p-1)/(p-q)}_\mu(\mathbb{R}^{1+n}_+)$$

via letting $\rho\to\infty$ and utilizing the monotone convergence theorem.

Step 2. We prove

$$P^R_{\alpha p}\mu \in L^{q(p-1)/(p-q)}_\mu(\mathbb{R}^{1+n}_+) \Rightarrow (1.2).$$

Recall that (1.2) is equivalent to the following inequality

$$\|R^*_\alpha(\mathbf{g}d\mu)\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|\mathbf{g}\|_{L^{q'}_\mu(\mathbb{R}^{1+n}_+)} \quad \forall\ \mathbf{g} \in L^{q'}_\mu(\mathbb{R}^{1+n}_+).$$

Thus, by Lemma 2.3, it is sufficient to check that $P^R_{\alpha p}\mu \in L^{q(p-1)/(p-q)}_\mu(\mathbb{R}^{1+n}_+)$ implies

$$\int_{\mathbb{R}^{1+n}_+} P^R_{\alpha p}(\mathbf{g}d\mu)(t,x)\mathbf{g}(t,x)d\mu \lesssim \|\mathbf{g}\|^{p'}_{L^{q'}(\mathbb{R}^{1+n}_+,\, d\mu)} \quad \forall\ \mathbf{g} \in L^{q'}_\mu(\mathbb{R}^{1+n}_+). \tag{2.5}$$

There is no loss of generality in assuming $\mathbf{g} \geq 0$. Since

$$\begin{aligned} P^R_{\alpha p}(\mathbf{g}d\mu)(t,x) &\approx \int_0^\infty \left(\frac{\mu(B^{(\alpha)}_r(t,x))}{r^n}\right)^{p'-1} \left(\frac{\int_{B^{(\alpha)}_r(t,x)} \mathbf{g}(t,x)d\mu}{\mu(B^{(\alpha)}_r(t,x))}\right)^{p'-1} \frac{dr}{r} \\ &\lesssim (M_\mu \mathbf{g}(t,x))^{p'-1} P^R_{\alpha p}\mu(t,x), \end{aligned}$$

an application of the Hölder inequality gives

$$\begin{aligned} &\int_{\mathbb{R}^{1+n}_+} P^R_{\alpha p}(\mathbf{g}d\mu)(t,x)d\mu(t,x) \\ &\lesssim \int_{\mathbb{R}^{1+n}_+} (M_\mu \mathbf{g}(t,x))^{p'-1} P^R_{\alpha p}\mu(t,x)\mathbf{g}(t,x)d\mu(t,x) \\ &\lesssim \left(\int_{\mathbb{R}^{1+n}_+} (M_\mu \mathbf{g}(t,x))^{q'} d\mu(t,x)\right)^{\frac{q'}{p'-1}} \left(\int_{\mathbb{R}^{1+n}_+} \left(\mathbf{g}(t,x)P^R_{\alpha p}\mu(t,x)\right)^{\frac{q'}{q'-p'+1}} d\mu(t,x)\right)^{\frac{q'-p'+1}{q'}}. \end{aligned}$$

Here

$$M_\mu \mathbf{g}(t,x) = \sup_{r>0} \frac{1}{\mu(B^{(\alpha)}_r(t,x))} \int_{B^{(\alpha)}_r(t,x)} \mathbf{g}(s,y)d\mu(s,y)$$

is the centered Hardy-Littlewood maximal function of $\mathbf{g}$ with respect to μ. The fact that M_μ is bounded on $L^{q'}_\mu(\mathbb{R}^{1+n}_+)$ (cf. [11]) and Hölder's inequality imply

$$\int_{\mathbb{R}^{1+n}_+} P^R_{\alpha p}(\mathbf{g}d\mu)(t,x)d\mu(t,x) \lesssim \|\mathbf{g}\|^{p'}_{L^{q'}_\mu(\mathbb{R}^{1+n}_+)} \left(\int_{\mathbb{R}^{1+n}_+} \left(P^R_{\alpha p}\mu\right)^{\frac{q(p-1)}{p-q}} d\mu\right)^{\frac{p-q}{q(p-1)}},$$

whence (2.5). □

3. S_α's tracing

In this section we verify Theorem 1.1 for $T_\alpha = S_\alpha$ and $1 < p < 1 + \frac{n}{2\alpha}$. Like proving Theorem 1.1 for $T_\alpha = R_\alpha$, three lemmas are required in what follows.

The first is regarding the dual formulation of $C^{(S_\alpha)}_p(K)$ of a given compact set $K \subset \mathbb{R}^{1+n}_+$.

Lemma 3.1. *Let $\mu \in \mathcal{U}^+(K)$, $1 < p < 1 + \frac{n}{2\alpha}$, $p' = \frac{p}{p-1}$, S^*_α be the adjoint operator of S_α, and*

$$P^S_{\alpha p}\mu(t,x) = \int_0^\infty \left(\frac{\mu\big(B^{(\alpha)}_r(t,x)\big)}{r^{n+2\alpha(1-p)}}\right)^{p'-1} \frac{dr}{r} \quad \forall \quad (t,x) \in \mathbb{R}^{1+n}_+.$$

Then:

(*a*)
$$C_p^{(S_\alpha)}(K) = \sup\{\|\mu\|^p : \mu \in \mathcal{U}^+(K) \ \&\ \|S_\alpha^*\mu\|_{L^{p'}(\mathbb{R}_+^{1+n})} \le 1\}.$$

(*b*)
$$\|S_\alpha^*\mu\|^{p'}_{L^{p'}(\mathbb{R}_+^{1+n})} \approx \int_{\mathbb{R}_+^{1+n}} P^S_{\alpha p}\mu(t,x)d\mu(t,x).$$

Proof. (a) Since S_α^* is determined by
$$\int_{\mathbb{R}_+^{1+n}} (S_\alpha g)h\,dtdx = \int_{\mathbb{R}_+^{1+n}} g(t,x)\left(\int_t^\infty e^{-(s-t)(-\Delta_x)^\alpha}h(s,x)ds\right)dtdx\ \forall\ g,h \in C_0^\infty(\mathbb{R}_+^{1+n}),$$
it follows that for any $h \in C_0^\infty(\mathbb{R}_+^{1+n})$ one has
$$S_\alpha^*h(t,x) = \int_t^\infty e^{-(s-t)(-\Delta_x)^\alpha}h(s,x)ds = \int_{[t,\infty)\times\mathbb{R}^n} K^{(\alpha)}_{s-t}(x-y)h(s,t)\,dsdy.$$
The definition of S_α^* is extended to the family of all Borel measures μ with compact support in $\mathbb{R}_+^{1+n}$:
$$S_\alpha^*\mu(t,x) = \int_{[t,\infty)\times\mathbb{R}^n} K^{(\alpha)}_{s-t}(x-y)\,d\mu(s,y).$$
According to [12, Propostion 2.1], we have
$$C_p^{(S_\alpha)}(K) = \sup\{\|\mu\|^p :\ \mu \in \mathcal{U}^+(K)\ \&\ \|S_\alpha^*\mu\|_{L^{p'}(\mathbb{R}_+^{1+n})} \le 1\}.$$

(b) This can be proved via a slight modification of the argument for Lemma 2.3 - in particular - via replacing the maximal function $M_\alpha\mu(x)$ by
$$M_\alpha\mu(t,x) = \sup_{r>0} r^{-n}\int_{B_r^{(\alpha)}(t,x)} d\mu.$$
□

The second indicates that $C_p^{(S_\alpha)}(K)$ of a given compact $K \subset \mathbb{R}_+^{1+n}$ can be realized by $\mu_K(K)$ of an element $\mu_K \in \mathcal{U}^+(K)$.

Lemma 3.2. *Let K be a compact subset of $\mathbb{R}_+^{1+n}$. Then there exists a $\mu_K \in \mathcal{U}^+(K)$ such that*
$$\mu_K(K) = \int_{\mathbb{R}_+^{1+n}} (S_\alpha^*\mu_K(t,x))^{p'}dtdx = \int_{\mathbb{R}_+^{1+n}} S_\alpha(S_\alpha^*\mu_K)^{p'-1}d\mu_K = C_p^{(S_\alpha)}(K).$$

Proof. Lemma 3.1(a) (plus [12, Propostion 2.1]) ensures the existence of a sequence $\{\mu_i\} \subset \mathcal{U}^+(K)$ such that
$$\|S_\alpha^*\mu_i\|_{L^{p'}(\mathbb{R}_+^{1+n})} \le 1 \ \ \&\ \ \lim_{i\to\infty}\mu_i(K) = \left(C_p^{(S_\alpha)}(K)\right)^{\frac{1}{p}}$$
and μ_i has a weak limit $\mu \in \mathcal{U}^+(K)$. Thus $\mu(K) = \left(C_p^{(S_\alpha)}(K)\right)^{\frac{1}{p}}$. It follows from the lower semi-continuity of $S_\alpha^*\mu$ on $\mathcal{U}^+(K)$ that $\|S_\alpha^*\mu\|_{L^{p'}(\mathbb{R}_+^{1+n})} \le 1$. Meanwhile, the following estimation
$$\|\mu\| \le \int_{\mathbb{R}_+^{1+n}} S_\alpha g d\mu = \int_{\mathbb{R}_+^{1+n}} g(t,x)S_\alpha^*\mu(t,x)dtdx \le \|g\|_{L^p(\mathbb{R}_+^{1+n})}\|S_\alpha^*\mu\|_{L^{p'}(\mathbb{R}_+^{1+n})}$$
gives $\|S_\alpha^*\mu\|_{L^{p'}(\mathbb{R}_+^{1+n})} \ge 1$. So, $\|S_\alpha^*\mu\|_{L^{p'}(\mathbb{R}_+^{1+n})} = 1$.

Choosing $\mu_K = C_p^{(S_\alpha)}(K)^{\frac{1}{p'}}\mu$ and using $\mu(K) = \big(C_p^{(S_\alpha)}(K)\big)^{\frac{1}{p}}$, one has

$$\mu_K(K) = \int_{\mathbb{R}^{1+n}_+} (S^*_\alpha \mu_K(t,x))^{p'} dtdx = C_p^{(S_\alpha)}(K).$$

Suppose that g_0 is the capacitary potential of $C_p^{(S_\alpha)}(K)$, i.e.,

$$\|g_0\|^p_{L^p(\mathbb{R}^{1+n}_+)} = C_p^{(S_\alpha)}(K) \ \& \ S_\alpha g_0 \geq \mathbf{1}_K.$$

Then $g_0(t,x) = (S^*_\alpha \mu_K)^{p'-1}(t,x)$. A further use of [12, Propostion 2.1] derives

$$\mu_K(\{(t,x) \in K : S_\alpha g_0(t,x) < 1\}) = 0,$$

whence

$$S_\alpha(g_0) = S_\alpha(S^*_\alpha \mu_K)^{p'-1} \geq 1 \ \text{ a.e. } \mu_K \text{ on } K.$$

Now, Fubini's theorem and the Hölder inequality are utilized to derive

$$\begin{aligned}
C_p^{(S_\alpha)}(K) &\leq \int_{\mathbb{R}^{1+n}_+} S_\alpha g_0 d\mu_K \\
&= \int_{\mathbb{R}^{1+n}_+} \int_0^t \int_{\mathbb{R}^n} K^{(\alpha)}_{t-s}(x-y) f(s,y) dyds d\mu_K \\
&= \int_{\mathbb{R}^n} \int_0^t \int_s^\infty \int_{\mathbb{R}^n} K^{(\alpha)}_{t-s}(x-y) d\mu_K f(s,y) dsdy \\
&\leq \int_{\mathbb{R}^{1+n}_+} S^*_\alpha \mu_K(t,x) g_0(t,x) dtdx \\
&\leq \|S^*_\alpha \mu_K\|_{L^{p'}(\mathbb{R}^{1+n}_+)} \|g_0\|_{L^p(\mathbb{R}^{1+n}_+)} \\
&= C_p^{(S_\alpha)}(K),
\end{aligned}$$

thereby completing the proof. □

The third is concerning the weak and strong type estimates for $C_p^{(S_\alpha)}$.

Lemma 3.3. *Let $1 < p < \infty$ and $L^p_+(\mathbb{R}^{1+n}_+)$ stand for the class of all nonnegative functions in $L^p(\mathbb{R}^{1+n}_+)$. If $g \in L^p_+(\mathbb{R}^{1+n}_+)$ and $\lambda > 0$, then:*

(a) $C_p^{(S_\alpha)}(\{(t,x) \in \mathbb{R}^{1+n}_+ : S_\alpha g(t,x) \geq \lambda\}) \leq \lambda^{-p} \|g\|^p_{L^p(\mathbb{R}^{1+n}_+)}$;

(b) $\int_0^\infty C_p^{(S_\alpha)}(\{(t,x) \in \mathbb{R}^{1+n}_+ : S_\alpha g(t,x) \geq \lambda\}) d\lambda^p \lesssim \|g\|^p_{L^p(\mathbb{R}^{1+n}_+)}$.

Proof. (a) This follows immediately from the definition of $C_p^{(S_\alpha)}$.

(b) It is enough to check this inequality for any nonnegative function $g \in C_0^\infty(\mathbb{R}^{1+n}_+)$. The forthcoming demonstration is a slight modification of the argument for [7, Lemma 3.1].

For each $i = 0, \pm 1, \pm 2, \cdots$ and any nonnegative function $g \in C_0^\infty(\mathbb{R}^{1+n}_+)$, we follow the proof of [2, Theorem 7.1.1] to write

$$K_i = \{(t,x) \in \mathbb{R}^{1+n}_+ : S_\alpha g(t,x) \geq 2^i\}.$$

Assume that μ_i is the measure obtained in Lemma 3.2 for K_i. Then by duality and Hölder's inequality, one has

$$\begin{aligned}\sum_{i=-\infty}^{\infty} 2^{ip}\mu_i(\mathbb{R}^{1+n}_+) &\le \sum_{i=-\infty}^{\infty} 2^{i(p-1)}\int_{\mathbb{R}^{1+n}_+} g(t,x)S^*_\alpha\mu_i(t,x)dtdx\\ &\lesssim \|g\|_{L^p(\mathbb{R}^{1+n}_+)}\left\|\sum_{i=-\infty}^{\infty} 2^{i(p-1)}S^*_\alpha\mu_i\right\|_{L^{p'}(\mathbb{R}^{1+n}_+)}\\ &\equiv \|g\|_{L^p(\mathbb{R}^{1+n}_+)}\|\eta\|_{L^{p'}(\mathbb{R}^{1+n}_+)},\end{aligned}$$

where

$$\eta(t,x)=\sum_{i=-\infty}^{\infty} 2^{i(p-1)}S^*_\alpha\mu_i(t,x).$$

For $k=0,\pm1,\pm2,\cdots$, let

$$\eta_k(t,x)=\sum_{i=-\infty}^{k} 2^{i(p-1)}S^*_\alpha\mu_i(t,x).$$

Then it is easy to find that

$$\eta_k\in L^{p'}(\mathbb{R}^{1+n}_+)\ \&\ \lim_{k\to\infty}\eta_k=\eta \text{ in } L^{p'}(\mathbb{R}^{1+n}_+).$$

We next to prove that

$$\|\eta\|^{p'}_{L^{p'}(\mathbb{R}^{1+n}_+)}\lesssim\sum_{i=-\infty}^{\infty} 2^{ip}\mu_i(\mathbb{R}^{1+n}_+) \tag{3.1}$$

according to two cases.

Case: $2<p<\infty$. Notice first that

$$\eta(t,x)^{p'}=p'\sum_{k=-\infty}^{\infty}\eta_k(t,x)^{p'-1}2^{k(p-1)}S^*_\alpha\mu_k(t,x)\quad\text{a.e. } (t,x)\in\mathbb{R}^{1+n}_+. \tag{3.2}$$

So, the Hölder inequality yields that

$$\|\eta\|^{p'}_{L^{p'}(\mathbb{R}^{1+n}_+)}\lesssim\frac{\left(\int_{\mathbb{R}^{1+n}_+}\sum_{k=-\infty}^{\infty}2^{kp}(S^*_\alpha\mu_k)^{p'}(t,x)dtdx\right)^{2-p'}}{\left(\int_{\mathbb{R}^{1+n}_+}\sum_{k=-\infty}^{\infty}\eta_k(t,x)2^k(S^*_\alpha\mu_k)^{p'-1}(t,x)dtdx\right)^{1-p'}}.$$

Since Lemma 3.2 gives

$$\int_{\mathbb{R}^{1+n}_+}\sum_{k=-\infty}^{\infty}2^{kp}(S^*_\alpha\mu_k)^{p'}(t,x)dtdx=\sum_{k=-\infty}^{\infty}2^{kp}\int_{\mathbb{R}^{1+n}_+}(S^*_\alpha\mu_k)^{p'}(t,x)dtdx=\sum_{k=-\infty}^{\infty}2^{kp}\mu_k(\mathbb{R}^{1+n}_+)$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^{1+n}_+}\sum_{k=-\infty}^{\infty}\eta_k(t,x)2^k(S^*_\alpha\mu_k)^{p'-1}(t,x)dtdx\\
&=\sum_{k=-\infty}^{\infty}\sum_{i\le k}2^{i(p-1)+k}\int_{\mathbb{R}^{1+n}_+}(S^*_\alpha\mu_i(t,x))(S^*_\alpha\mu_k(t,x))^{p'-1}dtdx\\
&\lesssim\sum_{k=-\infty}^{\infty}2^{kp}C_p^{(S_\alpha)}(K_k)\\
&\approx\sum_{k=-\infty}^{\infty}2^{kp}\mu_k(\mathbb{R}^{1+n}_+),
\end{aligned}
$$

(3.1) is true for $2<p<\infty$.

Case: $1<p\le 2$. A combination of (3.2) and Minkowski's inequality gives

$$
\begin{aligned}
\|\eta\|^{p'}_{L^{p'}(\mathbb{R}^{1+n}_+)}&=\sum_{k=-\infty}^{\infty}2^{k(p-1)}\int_{\mathbb{R}^{1+n}_+}\left(\sum_{i=-\infty}^{k}2^{i(p-1)}S^*_\alpha\mu_i(t,x)\right)^{p'-1}(S^*_\alpha\mu_k(t,x))dtdx\\
&\lesssim\sum_{k=-\infty}^{\infty}2^{k(p-1)}\left(\sum_{i=-\infty}^{k}2^{i(p-1)}\left(\int_{\mathbb{R}^{1+n}_+}\left(S^*_\alpha\mu_i(t,x)\right)^{p'-1}S^*_\alpha\mu_i(t,x)dtdx\right)^{\frac{1}{p'-1}}\right)^{p'-1}\\
&\approx\sum_{k=-\infty}^{\infty}2^{k(p-1)}\left(\sum_{i=-\infty}^{k}2^{i(p-1)}C_p^{(S_\alpha)}(K_i)^{\frac{1}{p'-1}}\right)^{p'-1}\\
&\lesssim\sum_{k=-\infty}^{\infty}2^{k(p-1)}C_p^{(S_\alpha)}(K_k)\\
&\lesssim\sum_{k=-\infty}^{\infty}2^{k(p-1)}\mu_k(\mathbb{R}^{1+n}_+),
\end{aligned}
$$

whence yields (3.1) under $1<p\le 2$.

As a consequence, (3.1) plus

$$\sum_{i=-\infty}^{\infty}2^{ip}\mu_i(\mathbb{R}^{1+n}_+)\lesssim\sum_{i=-\infty}^{\infty}2^{ip}C_p^{(S_\alpha)}(K_i)\lesssim\|g\|^p_{L^p(\mathbb{R}^{1+n}_+)},$$

implies the desired inequality in (b). □

Now, Theorem 1.1 for $T_\alpha=S_\alpha$ is contained in the following assertion.

Theorem 3.4. *For a nonnegative Radon measure μ on $\mathbb{R}^{1+n}_+$ and $\lambda>0$ set*

$$C_S(\mu;\lambda)=\inf\left\{C_p^{(S_\alpha)}(K):\ \text{compact } K\subset\mathbb{R}^{1+n}_+\ \&\ \mu(K)\ge\lambda\right\}.$$

(1) *If $1<p<\min\{q,1+\frac{n}{2\alpha}\}$ then*

$$(1.3)\Leftrightarrow\sup_{\lambda>0}\frac{\lambda^{\frac{p}{q}}}{C_S(\mu;\lambda)}<\infty\Leftrightarrow\sup_{(r,t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n}\frac{\mu(B_r^{(\alpha)}(t_0,x_0))}{r^{\frac{(n+2\alpha(1-p))q}{p}}}<\infty.$$

(2) *If* $1 < p = q < 1 + \frac{n}{2\alpha}$ *then*

$$(1.3) \Leftrightarrow \sup_{\lambda>0} \frac{\lambda}{C_S(\mu;\lambda)} < \infty \quad \left(\Rightarrow \sup_{(r,t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n} \frac{\mu(B_r^{(\alpha)}(t_0,x_0))}{r^{n+2\alpha(1-p)}} < \infty\right).$$

(3) $1 < q < p < 1 + \frac{n}{2\alpha}$ *then*

$$(1.3) \Leftrightarrow \int_0^\infty \left(\frac{\lambda^{\frac{p}{q}}}{C_S(\mu;\lambda)}\right)^{\frac{q}{p-q}} \frac{d\lambda}{\lambda} < \infty \Leftrightarrow P^S_{\alpha p}\mu \in L^{q(p-1)/(p-q)}_\mu(\mathbb{R}^{1+n}_+).$$

Proof. (1) Suppose (1.3) is valid. Then, for a given compact set $K \subset \mathbb{R}^{1+n}_+$, an application of Lemma 3.2 and Hölder's inequality gives

$$\int_{\mathbb{R}^{1+n}_+} g S^*_\alpha \mu_K dtdx = \int_{\mathbb{R}^{1+n}_+} S_\alpha g d\mu_K \le \|S_\alpha g\|_{L^q_\mu(\mathbb{R}^{1+n}_+)} \mu(K)^{\frac{1}{q'}} \lesssim \|g\|_{L^p(\mathbb{R}^{1+n}_+)} \mu(K)^{\frac{1}{q'}},$$

whence

$$\|S^*_\alpha \mu_K\|_{L^{p'}(\mathbb{R}^{1+n}_+)} \lesssim \mu(K)^{\frac{1}{q'}}.$$

This shows that for

$$E_\lambda(g) \equiv \left\{(t,x) \in \mathbb{R}^{1+n}_+ : |S_\alpha g(t,x)| \ge \lambda\right\} \quad \forall \quad \lambda > 0$$

one has

$$\begin{aligned} \lambda\mu(E_\lambda(g)) &\le \int_{\mathbb{R}^{1+n}_+} |S_\alpha g| d\mu_{E_\lambda} \\ &\lesssim \|g\|_{L^p(\mathbb{R}^{1+n}_+)} \|S^*_\alpha \mu_{E_\lambda}\|_{L^{p'}(\mathbb{R}^{1+n}_+)} \\ &\lesssim \|g\|_{L^p(\mathbb{R}^{1+n}_+)} \mu(E_\lambda)^{\frac{1}{q'}}. \end{aligned}$$

Therefore, we obtain

$$\sup_{\lambda>0} \lambda^q \mu(E_\lambda(g)) \lesssim \|g\|^q_{L^p(\mathbb{R}^{1+n}_+)}.$$

Picking a function $g \in L^p(\mathbb{R}^{1+n}_+)$ such that $S_\alpha g \ge 1$ on a given compact $K \subset \mathbb{R}^{1+n}_+$, we conclude that

$$\mu(K)^{\frac{1}{q}} \lesssim C_p^{(S_\alpha)}(K)^{\frac{1}{p}} \text{ and hence } \lambda^{\frac{1}{q}} \lesssim C_S(\mu;\lambda)^{\frac{1}{p}} \ \forall \ \lambda > 0.$$

Conversely, if the last inequality is valid, then

$$\mu(K)^{\frac{1}{q}} \lesssim C_p^{(S_\alpha)}(K)^{\frac{1}{p}} \ \ \forall \text{ compact } K \subset \mathbb{R}^{1+n}_+.$$

Lemma 3.3 is used to derive that if $g \in L^p(\mathbb{R}^{1+n}_+)$ then

$$\begin{aligned} \int_{\mathbb{R}^{1+n}_+} |S_\alpha g|^q d\mu &= \int_0^\infty \mu(E_\lambda) d\lambda^q \\ &\lesssim \int_0^\infty C_p^{(S_\alpha)}(E_\lambda)^{\frac{q-p}{p}} C_p^{(S_\alpha)}(E_\lambda) \lambda^{q-p} d\lambda^p \\ &\lesssim \|g\|^{q-p}_{L^p(\mathbb{R}^{1+n}_+)} \int_0^\infty C_p^{(S_\alpha)}(E_\lambda) d\lambda^p \\ &\lesssim \|g\|^q_{L^p(\mathbb{R}^{1+n}_+)}. \end{aligned}$$

Namely, (1.3) holds.

Next, an application of (1.5) derives that

$$\lambda^{\frac{1}{q}} \lesssim C_S(\mu;\lambda)^{\frac{1}{p}}\ \forall\ \lambda>0 \Rightarrow \mu(B_r^{(\alpha)}(t_0,x_0)) \lesssim r^{\frac{q}{p}(n+2\alpha-2\alpha p)}\quad \forall\ r>0.$$

For the reverse implication, we first note that $(t,x)\in B_r^{(\alpha)}(t_0,x_0)$ ensures $K_{t-t_0}^{(\alpha)}(x-x_0)\gtrsim r^{-n}$. This, along with Fubini's theorem, yields

$$\begin{aligned}
S_\alpha^*\mu_K(t_0,x_0) &\approx \int_{t_0}^\infty\int_{\mathbb{R}^n}\left(\int_{(K_{t-t_0}^{(\alpha)}(x-x_0))^{-\frac{1}{n}}}^\infty \frac{dr}{r^{n+1}}\right)d\mu_K\\
&\lesssim \int_{t_0}^\infty\int_{\mathbb{R}^n}\left(\int_0^\infty \mathbf{1}_{B_r^{(\alpha)}(t_0,x_0)}\frac{dr}{r^{n+1}}\right)d\mu_K\\
&\lesssim \int_0^\infty \mu_K(B_r^{(\alpha)}(t_0,x_0))\frac{dr}{r^{n+1}}.
\end{aligned}$$

Therefore, for a $\delta>0$ to be determined later, we use the Minkwoski inequality to get

$$\begin{aligned}
\|S_\alpha^*\mu_K\|_{L^{p'}(\mathbb{R}^{1+n}_+)} &\lesssim \int_{\mathbb{R}^{1+n}_+}\left(\int_0^\infty \mu_K(B_r^{(\alpha)}(t_0,x_0))\frac{dr}{r^{n+1}}\right)^{p'}dtdx\\
&\lesssim \int_0^\infty \|\mu_K(B_r^{(\alpha)}(\cdot,\cdot))\|_{L^{p'}(\mathbb{R}^{1+n}_+)}\frac{dr}{r^{n+1}}\\
&= \int_0^\delta \|\mu_K(B_r^{(\alpha)}(\cdot,\cdot))\|_{L^{p'}(\mathbb{R}^{1+n}_+)}\frac{dr}{r^{n+1}}\\
&\quad + \int_\delta^\infty \|\mu_K(B_r^{(\alpha)}(\cdot,\cdot))\|_{L^{p'}(\mathbb{R}^{1+n}_+)}\frac{dr}{r^{n+1}}\\
&\equiv I_1+I_2.
\end{aligned}$$

Since

$$\|\mu_K(B_r^{(\alpha)}(\cdot,\cdot))\|^{p'}_{L^{p'}(\mathbb{R}^{1+n}_+)} \lesssim \mu(K)^{p'-1}\int_{\mathbb{R}^{1+n}_+}\mu_K(B_r^{(\alpha)}(t,x))dtdx \lesssim \mu(K)^{p'-1}r^{n+2\alpha},$$

it follows that

$$I_2 \lesssim \mu(K)\int_\delta^\infty \frac{dr}{r^{n+1-\frac{n+2\alpha}{p'}}} \lesssim \mu(K)\delta^{2\alpha-\frac{n+2\alpha}{p}}.$$

On the other hand,

$$\mu(B_r^{(\alpha)}(t_0,x_0)) \lesssim r^{\frac{q}{p}(n+2\alpha-2\alpha p)}\ \forall\ r>0$$

derives

$$\begin{aligned}
\|\mu_K(B_r^{(\alpha)}(\cdot,\cdot))\|^{p'}_{L^{p'}(\mathbb{R}^{1+n}_+)} &\lesssim r^{\frac{q(n+2\alpha-2\alpha p)(p'-1)}{p}}\int_{\mathbb{R}^{1+n}_+}\mu_K(B_r^{(\alpha)}(t,x))dtdx\\
&\lesssim \mu(K)r^{\frac{q(n+2\alpha-2\alpha p)(p'-1)}{p}+n+2\alpha}.
\end{aligned}$$

This clearly forces

$$I_1 \lesssim \mu(K)^{\frac{1}{p'}}\int_0^\delta r^{(p')^{-1}\left(\frac{q(n+2\alpha-2\alpha p)(p'-1)}{p}+n+2\alpha\right)}\frac{dr}{r^{n+1}} \lesssim \mu(K)^{\frac{1}{p'}}\delta^{\frac{(q-p)(n+2\alpha-2\alpha p)}{p^2}}.$$

Upon choosing $\delta=\mu(K)^{\frac{p}{q(n+2\alpha-2\alpha p)}}$, we obtain

$$\|S_\alpha^*\mu_K\|_{L^{p'}(\mathbb{R}^{1+n}_+)}\lesssim\mu(K)^{\frac{1}{q'}}\text{ and hence }C_p^{(S_\alpha)}(K)^{\frac{1}{p'}}\lesssim\mu(K)^{\frac{1}{q'}}.$$

(2) This follows from the above demonstration.

(3) Suppose (1.3) is valid. Then

$$\sup_{\lambda>0}\lambda(\mu(E_\lambda(g)))^{\frac{1}{q}}\lesssim\|g\|_{L^p(\mathbb{R}^{1+n}_+)}\ \ \forall\, g\in L^p(\mathbb{R}^{1+n}_+).$$

For each integer $i\in\mathbb{Z}$, there is a compact set $K_i\subset\mathbb{R}^{1+n}_+$ and a function $g_i\in L^p(\mathbb{R}^{1+n}_+)$ such that

$$C_p^{(S_\alpha)}(K_i)\lesssim C_S(\mu;2^i),\ \ \mu(K_i)>2^i;\ \ S_\alpha g_i\geq\mathbf{1}_{K_i};\ \ \ \|g_i\|^p_{L^p(\mathbb{R}^{1+n}_+)}\lesssim C_p^{(S_\alpha)}(K_i).$$

Set

$$g_{j,k}=\sup_{j\leq i\leq k}\left(\frac{2^i}{C_S(\mu;2^i)}\right)^{\frac{1}{p-q}}g_i$$

for integers j,k with $j<k$. Then

$$\|g_{j,k}\|^p_{L^p(\mathbb{R}^{1+n}_+)}\lesssim\sum_{i=j}^k\left(\frac{2^i}{C_S(\mu;2^i)}\right)^{\frac{p}{p-q}}\|g_i\|^p_{L^p(\mathbb{R}^{1+n}_+)}\lesssim\sum_{i=j}^k\left(\frac{2^i}{C_S(\mu;2^i)}\right)^{\frac{p}{p-q}}C_S(\mu;2^i).$$

Since for $\forall\,(t,x)\in K_i$ and $j\leq i\leq k$ one has

$$|S_\alpha g_{j,k}(t,x)|\geq\left(\frac{2^i}{C_S(\mu;2^i)}\right)^{\frac{1}{p-q}}S_\alpha g_i(t,x)\gtrsim\left(\frac{2^i}{C_S(\mu;2^i)}\right)^{\frac{1}{p-q}},$$

it follows that

$$2^i<\mu(K_i)\leq\mu\left(E_{\left(\frac{2^i}{C_S(\mu;2^i)}\right)^{\frac{1}{p-q}}}(g_{j,k})\right),$$

and so that

$$\begin{aligned}\|g_{j,k}\|^q_{L^p(\mathbb{R}^{1+n}_+)}&\gtrsim\int_{\mathbb{R}^{1+n}_+}|S_\alpha g_{j,k}|^q d\mu\\&\gtrsim\sum_{i=j}^k\left(\frac{2^i}{C_S(\mu;2^i)}\right)^{\frac{q}{p-q}}2^i\\&\gtrsim\frac{\sum_{i=j}^k\left(\frac{2^i}{C_S(\mu;2^i)}\right)^{\frac{q}{p-q}}2^i\|g_{j,k}\|^q_{L^p(\mathbb{R}^{1+n}_+)}}{\left(\sum_{i=j}^k\left(\frac{2^i}{C_S(\mu;2^i)}\right)^{\frac{q}{p-q}}C_S(\mu;2^i)\right)^{\frac{q}{p}}}\\&\approx\left(\sum_{i=j}^k\frac{2^{\frac{ip}{p-q}}}{(C_S(\mu;2^i))^{\frac{q}{p-q}}}\right)^{\frac{p-q}{p}}\|g_{j,k}\|^q_{L^p(\mathbb{R}^{1+n}_+)}.\end{aligned}$$

This is the desired result thanks to

$$\int_0^\infty\left(\frac{\lambda^{\frac{p}{q}}}{C_S(\mu;\lambda)}\right)^{\frac{q}{p-q}}\frac{d\lambda}{\lambda}\lesssim\sum_{i=-\infty}^\infty\frac{2^{\frac{ip}{p-q}}}{(C_S(\mu;2^i))^{\frac{q}{p-q}}}\lesssim 1.$$

Conversely, if

$$\int_0^\infty \left(\frac{\lambda^{\frac{p}{q}}}{C_S(\mu;\lambda)}\right)^{\frac{q}{p-q}} \frac{d\lambda}{\lambda} < \infty,$$

then setting

$$T_{p,q}(\mu;g) = \sum_{i=-\infty}^{\infty} \frac{\left(\mu(E_{2^i}(g)) - \mu(E_{2^{i+1}}(g))\right)^{\frac{p}{p-q}}}{\left(C_p^{(S_\alpha)}(E_{2^i}(g))\right)^{\frac{q}{p-q}}}$$

for each integer $i = 0, \pm1, \pm2, \cdots$, and $g \in C_0^\infty(\mathbb{R}^{1+n}_+)$, we use an integration-by-part, the Hölder inequality and Lemma 3.3 to produce

$$\begin{aligned}
\int_{\mathbb{R}^{1+n}_+} |S_\alpha g|^q d\mu &= -\int_0^\infty \lambda^q d\mu(E_\lambda(g)) \\
&\lesssim \sum_{i=-\infty}^{\infty} \left(\mu(E_{2^i}(g)) - \mu(E_{2^{i+1}}(g))\right) 2^{iq} \\
&\lesssim \left(T_{p,q}(\mu;g)\right)^{\frac{p-q}{p}} \left(\sum_{i=-\infty}^{\infty} 2^{ip} C_p^{(S_\alpha)}\big(E_{2^i}(g)\big)\right)^{\frac{q}{p}} \\
&\lesssim \left(T_{p,q}(\mu;g)\right)^{\frac{p-q}{p}} \left(\int_0^\infty C_p^{(S_\alpha)}(\{(t,x) \in \mathbb{R}^{1+n}_+ : |S_\alpha g(t,x)| > \lambda\}) d\lambda^p\right)^{\frac{q}{p}} \\
&\lesssim \left(T_{p,q}(\mu;g)\right)^{\frac{p-q}{p}} \|g\|^q_{L^p(\mathbb{R}^{1+n}_+)} \\
&\lesssim \|g\|^q_{L^p(\mathbb{R}^{1+n}_+)}.
\end{aligned}$$

In the last inequality we have used the following estimation:

$$\begin{aligned}
\left(T_{p,q}(\mu;g)\right)^{\frac{p-q}{p}} &\lesssim \left(\sum_{i=-\infty}^{\infty} \frac{\left(\mu(E_{2^i}(g)) - \mu(E_{2^{i+1}}(g))\right)^{\frac{p}{p-q}}}{\left(C_S(\mu;\mu(E_{2^i}(g)))\right)^{\frac{q}{p-q}}}\right)^{\frac{p-q}{p}} \\
&\lesssim \left(\sum_{i=-\infty}^{\infty} \frac{\left(\mu(E_{2^i}(g))\right)^{\frac{p}{p-q}} - \left(\mu(E_{2^{i+1}}(g))\right)^{\frac{p}{p-q}}}{\left(C_S(\mu;\mu(E_{2^i}(g)))\right)^{\frac{q}{p-q}}}\right)^{\frac{p-q}{p}} \\
&\lesssim \left(\int_0^\infty \frac{ds^{\frac{p}{p-q}}}{\left(C_S(\mu;s)\right)^{\frac{q}{p-q}}}\right)^{\frac{p-q}{p}} \\
&\approx \left(\int_0^\infty \left(\frac{\lambda^{\frac{q}{p}}}{C_S(\mu;s)}\right)^{\frac{q}{p-q}} \frac{d\lambda}{\lambda}\right)^{\frac{p-q}{p}}.
\end{aligned}$$

Needless to say, the equivalence

$$(1.3) \Leftrightarrow P^S_{\alpha p}\mu \in L^{q(p-1)/(p-q)}_\mu(\mathbb{R}^{1+n}_+)$$

follows from Lemma 3.1(b) and a modification (cf. [6, Theorem 2.1]) of the argument for

$$(1.2) \Leftrightarrow P^R_{\alpha p}\mu \in L^{q(p-1)/(p-q)}_\mu(\mathbb{R}^{1+n}_+),$$

and hence the interested reader can readily work out the details. □

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